

We Shall Know: Hilbert's Apology

This note is about a poem to the glory of Mathematics.

On January 23, 1930, David Hilbert reached the mandatory retirement age of 68. Among the many honours bestowed upon him, he was made an "honorary citizen" (*Ehrenbürger*) of his native town of Königsberg. The honorary citizenship was presented to Hilbert on September 8, 1930, at the meeting of the Society of German Scientists and Physicians which was held that year in Königsberg. Hilbert's acceptance address was entitled "Natural Philosophy and Logic" (*Naturerkennen und Logik*).¹

Though to the outside world his name was inextricably linked to the scientific glory of Göttingen, where he was called on Klein's initiative in 1895, Hilbert himself had always remained deeply attached to the town of his forefathers. It was in Königsberg that he had started his career, and formed a lifelong friendship, mathematical as well as personal, with Hurwitz and Minkowski. He was very pleased to be honoured by the town of Königsberg on his retirement year. It was a perfect place and occasion to express, towards the end of his mathematical life, his views on the meaning and the goals of mathematics, and of science in general. Hilbert was glad to address a general scientific audience rather than just professional mathematicians. Already in his talk [6] at the International Congress of Mathematicians at Paris in 1900 he had said, quoting an unnamed old French mathematician, "A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street."

Reidemeister and Szegő made arrangements for Hilbert to repeat a précis of the last part of his acceptance address at the Society meeting over the local radio station. A 45-

rpm-record of this radio broadcast exists and was distributed with Hilbert's "Gedenkband" [9].

While the complete address at the Society meeting has been duly published [7] (though never translated to English except for selected passages in Constance Reid's book [12, Chapter 22]), the text of the radio broadcast has never been printed. Even though it is essentially extracted from the complete address, the text of the radio broadcast has a life and a meaning of its own, a short and concise statement on the goals and the place of mathematics quite different from the detailed discussion in [7]. I first received a transcript of Hilbert's recording from my colleague Victor Katsnelson about two years ago, while on a sabbatical at MSRI in Berkeley. On my suggestion the text was translated to English by Joseph and Amelia Ball, and distributed among the happy few at MSRI and elsewhere. Everybody—including occasional non-mathematicians—was impressed by the force of Hilbert's message; maybe we felt we were hearing "from afar" (as Hermann Weyl wrote in his reminiscences) "the sweet flute of the Pied Piper . . . seducing so many rats to follow him into the deep river of mathematics" [18]. Soon another translation was made by Laurent Siebenmann. The translation below is the result of joint efforts of Joseph and Amelia Ball and Laurent Siebenmann, some very helpful remarks by Otto Siebenmann, and small adjustments by myself. I am very glad of this opportunity to present anew this powerful apology of our science.

Das Instrument, welches die Vermittlung bewirkt zwischen Theorie und Praxis, zwischen Denken und Beobachten, ist die Mathematik; sie baut die verbind-

¹The literal translation of the German *Naturerkennen* is *knowledge (or understanding) of nature*. But I think that the meaning is *Natural Philosophy*, in its older English sense.

dende Brücke und gestaltet sie immer tragfähiger. Daher kommt es, daß unsere ganze gegenwärtige Kultur, soweit sie auf der geistigen Durchdringung und Dienstbarmachung der Natur beruht, ihre Grundlagen in der Mathematik findet. Schon Galilei sagt: Die Natur kann nur der verstehen, der ihre Sprache und die Zeichen kennengelernt hat, in der sie zu uns redet; diese Sprache aber ist die Mathematik, und ihre Zeichen sind die mathematischen Figuren. Kant tat den Ausspruch: "Ich behaupte, daß in jeder besonderen Naturwissenschaft nur soviel eigentliche Wissenschaft angetroffen werden kann, als darin Mathematik enthalten ist." In der Tat: Wir beherrschen nicht eher eine naturwissenschaftliche Theorie, als bis wir ihren mathematischen Kern herausgeschält und völlig enthüllt haben. Ohne Mathematik ist die heutige Astronomie und Physik unmöglich; diese Wissenschaften lösen sich in ihren theoretischen Teilen geradezu in Mathematik auf. Diese wie die zahlreichen weiteren Anwendungen sind es, den die Mathematik ihr Ansehen verdankt, soweit sie solches im weiteren Publikum genießt.²

The tool that serves as intermediary between theory and practice, between thought and observation, is mathematics; it is mathematics that builds the linking bridges, and gives them ever more reliable forms. From this it has come about that our entire contemporary culture, inasmuch as it is based on the intellectual penetration and the exploitation of nature, has its foundations in mathematics. Already Galileo said: one can understand nature only when one has learned the language and the signs with which it speaks to us; but this language is mathematics and these signs are mathematical figures. Kant made the pronouncement: "I assert that, in any particular natural science, one encounters genuine scientific substance only to the extent that mathematics is present." Indeed: we do not master a scientific theory until we have shelled and completely pried free its mathematical kernel. Without mathematics, the astronomy and physics of today would be impossible; these sciences, in their theoretical branches, virtually dissolve into mathematics. They, along with the many other applications,

are responsible for whatever esteem mathematics may enjoy in the eyes of the general public.

Trotzdem haben es alle Mathematiker abgelehnt, die Anwendungen als Wertmesser für die Mathematik gelten zu lassen. Gauß spricht von dem zauberischen Reiz, den³ die Zahlentheorie zur Lieblingswissenschaft der ersten Mathematiker gemacht habe, ihres unerschöpflichen Reichtums nicht zu gedenken, woran sie alle anderen Teile der Mathematik so weit übertrifft. Kronecker vergleicht die Zahlentheoretiker mit den Lotophagen, die, wenn sie einmal von dieser Kost etwas zu sich genommen haben, nie mehr davon lassen können.

Nevertheless, all mathematicians have refused to accept the applications as a valid measure of the worth of mathematics. Gauss speaks of the magical attraction that made number theory the darling discipline of the earliest mathematicians, not to mention number theory's inexhaustible wealth, so far surpassing that of any other branch of mathematics. Kronecker compares number-theorists with the lotus eaters, who, once they have savored this delight, can never give it up.

Der große Mathematiker Poincaré wendet sich einmal in auffallender Schärfe gegen Tolstoi, der erklärt hatte, daß die Forderung "die Wissenschaft der Wissenschaft wegen" töricht sei.⁴ Die Errungenschaften der Industrie zum Beispiel hätten nie das Licht der Welt erblickt, wenn die Praktiker allein existiert hätten und wenn diese Errungenschaften nicht von uninteressierten Toren gefördert worden wären.⁵

The great mathematician Poincaré replied with remarkable sharpness to Tolstoy who had declared the pursuit of "Science for the sake of Science" to be foolish. The achievements of industry, for example, would never have seen the light of day, if only practical men had existed, and if these achievements had not been made possible by disinterested "fools." Die Ehre des menschlichen Geistes, so sagt der berühmte Königsberger Mathematiker Jacobi, ist der einzige Zweck aller Wissenschaft.⁶

The glory of the human spirit,⁷ so said the famous Königsberg mathematician Jacobi, is the one purpose of all science.

²In [7] this sentence is slightly different: "Diese und die zahlreichen weiteren Anwendungen sind es, denen die Mathematik ihr Ansehen verdankt, soweit sie solches im weiteren Publikum genießt."

³"der" in [7]; did Hilbert make a mistake in reading?

⁴In [7] this sentence starts simply with "Poincaré . . ."; there is a sentence preceding it, omitted in the radio broadcast, referring to Poincaré as "der glänzendste Mathematiker seiner Generation".

⁵In [7]: "Man braucht nur die Augen zu öffnen, so Schließt Poincaré, um zu sehen, wie zum Beispiel die Errungenschaften der Industrie nie das Licht der Welt erblickt hätten, wenn die Praktiker allein existiert hätten und wenn diese Errungenschaften nicht von uninteressierten Toren gefördert worden wären, die nie an die praktische Ausnützung gedacht haben."

⁶In [7] Hilbert refers to an argument between "unser großer Königsberger Mathematiker Jacobi" and "der berühmte Fourier"; Fourier was claiming that the purpose of mathematics lies in the explanation of natural phenomena. "Ein Philosoph, wie Fourier es doch sei, hätte wissen sollen, so ruft Jacobi, daß die Ehre des menschlichen Geistes der einzige Zweck aller Wissenschaft ist und daß unter diesem Gesichtspunkt ein Problem der reinen Zahlentheorie ebensoviel wert ist als eines, das den Anwendungen dient."

⁷Die Ehre des menschlichen Geistes" is not easy to translate to English. "The honour of the human spirit" or "The glory of the human intellect" are also possible translations.

Wir dürfen nicht denen glauben, die heute mit philosophischer Miene und überlegenem Tone den Kulturuntergang prophezeien und sich in dem Ignorabimus gefallen.⁸ Für uns⁹ gibt es kein Ignorabimus, und meiner Meinung nach auch für die Naturwissenschaft überhaupt nicht. Statt des törichten Ignorabimus heiße im Gegenteil unsere Lösung: **Wir müssen wissen—wir werden wissen.**

*We ought not to believe those who today, adopting a philosophical air and a tone of superiority, prophesy the decline of culture and are content with the “unknowable”¹⁰ in a self-satisfied way. For us there is no unknowable, and in my opinion there is also none whatsoever for the natural sciences. In place of this foolish “unknowable,” let our watchword on the contrary be: **We must know—we shall know.***

Beyond the sheer poetic power, Hilbert has addressed the fundamental issues facing all of us mathematicians, or at least those among us who (to cite Hermann Weyl again) “are not indifferent to what their scientific endeavours mean in the context of man’s whole caring and knowing, suffering and creative existence in the world” [19]. Is “We Shall Know!” still a valid motto for mathematics in the late 1990’s? It is impossible to present Hilbert’s apology of our science without saying something about one’s opinion.

To put the question in a proper perspective, let me recall that in the same fall 1930, on November 17, Kurt Gödel’s paper [4] containing the famous incompleteness theorems was submitted for publication. Gödel proved (the first incompleteness theorem) that given any consistent mathematical system $\mathfrak{F}$ containing elementary number theory, we can (effectively) construct a statement α in $\mathfrak{F}$ such that neither α nor the negation of α is provable in $\mathfrak{F}$.¹¹ Furthermore (the second incompleteness theorem), the consistency of $\mathfrak{F}$ cannot be proved in $\mathfrak{F}$.¹²

The discovery of paradoxes in set theory around the turn of the century had led to a “foundational crisis” in mathematics and a re-evaluation of standard mathematical arguments. To avoid the known paradoxes, Russell had introduced his theory of types; with perhaps less logical insight into the process of set formation, but in a way much more

suited to the usual mathematical practice, Zermelo, Fraenkel, von Neumann, and others had introduced a system of axioms for set theory which is now in common use under the name of ZFC (Zermelo–Fraenkel axiom system plus the axiom of choice). However, this gave no guarantee that no further paradoxes will appear. L. E. J. Brouwer proposed a much more radical reform with his intuitionism. As Weyl, Brouwer’s follower at the time, wrote later, “Brouwer opened our eyes and made us see how far classical mathematics, nourished by a belief in the ‘absolute’ that transcends all human possibilities of realization, goes beyond such statements as can claim real meaning and truth founded on evidence” [19]. Brouwer’s intuitionism accepts as valid only those mathematical arguments which provide an effective construction; in particular, intuitionism rejects the law of the excluded middle when applied to variables ranging over infinite domains. Only a limited part of classical mathematics can be reconstructed within intuitionism, and usually in a much more cumbersome way.¹³ Hilbert, for whom mathematical knowledge came before the foundational or philosophical issues underlying it, however important they were, could not accept this “mutilation” of science.¹⁴ But Hilbert also could not disregard Brouwer’s criticism, or the paradoxes of set theory.

Hilbert’s response was first to replace mathematical theories (elementary number theory, mathematical analysis, and eventually set theory as axiomatized in ZFC) by corresponding formal systems, reducing mathematical arguments, via the formalized rules of inference, to “formula games.” Then it is required to prove by *finitary* means (even more restricted than those of Brouwer) the consistency of these formal systems. This would clarify completely the nature of the foundations of mathematics and solve the foundational crisis once and for all without sacrificing any meaningful mathematical results. There is little doubt that when Hilbert said in his Königsberg address that in mathematics there is no unknowable, he meant, among other things, to assert his faith in the ultimate success of this programme.

Now, though Hilbert never specified precisely what finitary proofs were, they are definitely more restricted than proofs in elementary number theory. Hence Gödel’s sec-

⁸In [7]: “Wer die Wahrheit der großzügigen Denkweise und Weltanschauung, die aus diesen Worten Jacobis hervorleuchtet, empfindet, der verfällt nicht rückschrittlicher und unfruchtbarer Zweifelschut; der wird nicht denen glauben, . . .

⁹In [7] Hilbert writes “Für den Mathematiker” instead of “Für uns”.

¹⁰“Ignorabimus” in the original—a reference to “Ignoramus et ignorabimus” (“We are and we shall remain ignorant”) of the physiologist and philosopher Emile DuBois-Raymond (1818–1896) and his followers. DuBois-Raymond was claiming that there are predetermined limits to human knowledge, hence there are certain problems that the human intellect cannot solve even in principle, among these the nature of matter and force, the origin of motion, the origin of sensation and consciousness.

¹¹A mathematical system here means a formal system in the sense of mathematical logic, and a statement in $\mathfrak{F}$ means a (well formed) formula in $\mathfrak{F}$. It is assumed that $\mathfrak{F}$ has a finite number of symbols, rules of inference, and axiom schemes. Recall that (as one would expect) $\mathfrak{F}$ is called (simply) consistent if there is no formula α in $\mathfrak{F}$ such that both α and the negation of α are provable in $\mathfrak{F}$; it is intuitively obvious, and easy to prove rigorously, that if $\mathfrak{F}$ is not consistent, then any formula in $\mathfrak{F}$ is provable in $\mathfrak{F}$. In his original paper Gödel established the first incompleteness theorem under the additional assumption of the so-called ω -consistency of $\mathfrak{F}$, but it was later shown by Rosser [14] that simple consistency of $\mathfrak{F}$ is enough.

¹²A little more precisely, the consistency of $\mathfrak{F}$ can be expressed by a formula in $\mathfrak{F}$, but this formula cannot be proved in $\mathfrak{F}$.

¹³Though when available the intuitionistic proof often gives more information.

¹⁴Hilbert’s remark after Brouwer’s talk to the Mathematics Club at Göttingen in 1927, as reported by Reid [12, Chapter 21], is very apt: “With your methods, most of the results of modern mathematics would have to be abandoned, and to me the important thing is not to get fewer results but to get more results.” This also expresses well the reasons for the eventual rejection of the intuitionism by the vast majority of the mathematical community.

ond incompleteness theorem implies that Hilbert's programme, at least as originally conceived, is unfeasible. Furthermore, one may interpret the first incompleteness theorem as imposing certain *apriori* limits on mathematical knowledge. It may even be tempting to look down on Hilbert's belief in the power of the human intellect as somewhat naïve, akin to the widespread belief, in the beginning of the century, in the progress of civilization as the ultimate solution for all the problems of mankind.

However, such an attitude is superficial and does not do justice to the real value of either Gödel's incompleteness theorems or Hilbert's penetrating vision of the main driving force of the scientific enterprise—our quest for knowledge, and our certitude in the ultimate possibility of discovering it. The undecidable is *not* the unknowable, though Hilbert in his fondness for the axiomatic method and in his desire to resolve the foundational crisis may have somewhat confused the two. But in his talk [6] at Paris in 1900 he had said:

Occasionally it happens that we seek the solution [of a mathematical problem] under insufficient hypotheses or in an incorrect sense, and for this reason do not succeed. The problem then arises: to show the impossibility of the solution under the given hypotheses, or in the sense contemplated. Such proofs of impossibility were effected by the ancients, for instance when they showed that the ratio of the hypotenuse to the side of an isosceles right triangle is irrational. In later mathematics, the question as to the impossibility of certain solutions plays a preeminent part, and we perceive in this way that old and difficult problems, such as the proof of the axiom of parallels, the squaring of the circle, or the solution of equations of the fifth degree by radicals have finally found satisfactory and rigorous solutions, although in another sense than that originally intended. It is probably this important fact along with other philosophical reasons that gives rise to the conviction (which every mathematician shares, but which no one has as yet supported by a proof) that every definite mathematical problem must necessarily be susceptible of an exact settlement, either in the form of an actual answer to the question asked, or by the proof of the impossibility of its solution and therewith the necessary failure of all attempts.

The "impossibility results" that Hilbert has mentioned are related to some of the most striking advances in mathematics: the discovery of irrational and transcendental numbers, noneuclidean geometries, and Galois theory. Similarly, Gödel's incompleteness theorems have revolutionized mathematical logic and led to a subsequent deep and thorough study of (un)decidability, (un)solvability, and (un)computability.

Rather than refuting Hilbert's "We must know—we shall

know!", Gödel's theorems are a magnificent example of Hilbert's motto in action—one of the most outstanding achievements of mathematical knowledge in this century. I shall restrain myself from referring to other brilliant breakthroughs in the last 70 years, or even just the last decade.

Gödel's results and their subsequent development vindicate Hilbert on yet another point made in the address. While motivated originally only by our quintessential quest for knowledge, the theories of decidability, solvability, and computability became an essential component in modern-day computer science. There is no better reminder to our public leaders, senators, and (alas) many university presidents¹⁵ who accuse us, in the worst Tolstoyan way, of mispending public funds: our species would be still living in caves and/or jumping from tree to tree if not for this quest, pure and untainted. The fact that this quest eventually leads to the most utilitarian benefits in our lives may or may not be a fundamental law of nature, or a part of Divine Predestination—but it is a fact all the same.

For Some Further Reading

Constance Reid's biography [12] is the best source of details on Hilbert's personal and scientific life; this fascinating book should be read not only by any mathematician, but by anybody interested in the human creative process. There is a detailed account of the Königsberg address in Chapter 22.

There is a very good general survey of Hilbert's work, including the work on axiomatics and the foundations of mathematics, in Hermann Weyl's obituary article [18]. I recommend also Weyl's paper [19] for a veteran's account of the battles of the foundational crisis.

For a detailed account of Gödel's theorems and their place in mathematical logic and the foundations of mathematics see, e.g., Kleene's books [10, 11]. A good short introduction may be found in [5, Chapter 39]. Gödel's original paper [4] as well as an early survey by Rosser [15] still make a good reading; these two together with other foundational papers on decidability, solvability and computability have been conveniently translated and reprinted in Davis's anthology [2].

Even though Hilbert's programme cannot provide a complete and final answer for the problems of foundations of mathematics, it remains an extremely interesting research topic. See, e.g., [16, 17, 3].

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¹⁵If only they would bear in mind, when turning on their desktops and laptops, that computer-inventor John von Neumann was, for all his appetite for applications, a mathematician with a strong abstract bent.

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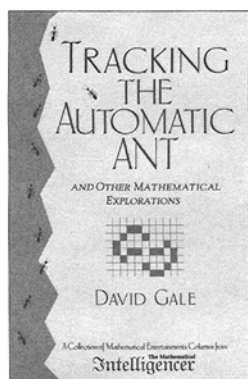
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